

# A NOTE ON LOCALLY CONNECTED AND HAMILTONIAN-CONNECTED GRAPHS

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## ABSTRACT

A graph  $G$  is locally  $n$ -connected,  $n \geq 1$ , if the subgraph induced by the neighborhood of each vertex is  $n$ -connected. It is shown that every connected, locally 3-connected graph containing no induced subgraph isomorphic to  $K(1, 3)$  is hamiltonian-connected.

## Introduction

In what follows let  $G$  denote a graph, as defined in [1]. If  $W$  is a nonempty subset of the vertex set  $V(G)$  of  $G$ , then we denote by  $\langle W \rangle$  the subgraph of  $G$  induced by  $W$ . For  $u \in V(G)$  the neighborhood  $N(u)$  of  $u$  is the set of all vertices of  $G$  adjacent to  $u$ . The graph  $G$  is locally  $n$ -connected,  $n \geq 1$ , if  $\langle N(u) \rangle$  is  $n$ -connected for each  $u \in V(G)$ . The graph  $G$  is hamiltonian-connected if for each pair of distinct vertices  $u$  and  $v$  of  $G$ , there is a hamiltonian  $u$ - $v$  path (that is, a  $u$ - $v$  path containing all vertices of  $G$ ).

Various articles have been written about hamiltonian properties of graphs which are connected, locally connected, and satisfy some additional, suitably chosen hypothesis. As illustrations we state two theorems, the first of which was proved by Chartrand and Pippert [1] and the second of which was proved by Oberly and Sumner [3].

**THEOREM A.** *If  $G$  is a connected, locally connected graph whose maximum degree  $\Delta(G)$  is at most 4, then  $G$  is either hamiltonian or isomorphic to  $K(1, 1, 3)$ .*

**THEOREM B.** *If  $G$  is a connected, locally connected graph of order  $p \geq 3$  which contains no induced subgraph isomorphic to  $K(1, 3)$ , then  $G$  is hamiltonian.*

In this note we are interested in when hypotheses similar to those used by Oberly and Sumner will imply that a graph is hamiltonian-connected. In

Received December 14, 1978

particular, we shall prove that if a graph  $G$  is connected, locally 3-connected, and contains no induced subgraph isomorphic to  $K(1, 3)$ , then  $G$  is hamiltonian-connected.

### Main Theorem

The following lemma, which is proved in [2], will be helpful in the ensuing arguments. We include its proof since it is not difficult and may help to clarify the concepts involved.

**LEMMA 1.** *If  $G$  is a connected, locally  $n$ -connected graph,  $n \geq 1$ , then  $G$  is  $(n + 1)$ -connected.*

**PROOF.** We proceed by induction on  $n$ . Let  $G$  be a graph which is connected and locally connected and assume  $G$  is not 2-connected. Then for some  $u \in V(G)$ , the graph  $G - u$  is disconnected, say  $G - u$  has components  $C_1, C_2, \dots, C_r$  ( $r \geq 2$ ). Then, in  $G$ , the vertex  $u$  is adjacent to at least one vertex of  $C_i$ ,  $1 \leq i \leq r$ . It follows that  $\langle N(u) \rangle$  is disconnected, a contradiction.

Next assume that if a graph is connected and locally  $k$ -connected for some  $k \geq 1$ , then it is  $(k + 1)$ -connected. Let  $G$  be a graph which is connected and locally  $(k + 1)$ -connected and let  $w \in V(G)$ . Then  $G - w$  is connected and is also locally  $k$ -connected. By the induction hypothesis,  $G - w$  is  $(k + 1)$ -connected. Thus  $G$  is  $(k + 2)$ -connected and the result follows. ■

The technique used in the proof of Theorem 1 borrows heavily from that used by Oberly and Sumner in their proof of Theorem B.

**THEOREM 1.** *If  $G$  is a connected, locally 3-connected graph which contains no induced subgraph isomorphic to  $K(1, 3)$ , then  $G$  is hamiltonian-connected.*

**PROOF.** We note that by Lemma 1,  $G$  is a 4-connected graph (and hence  $l$ -connected for  $1 \leq l \leq 4$ ). We proceed by contradiction and assume there is a graph  $G$  satisfying the hypothesis of the theorem but which is not hamiltonian-connected. Thus there exist distinct vertices  $u$  and  $v$  of  $G$  such that  $G$  contains no  $u$ - $v$  hamiltonian path. Since  $G$  is 2-connected there are at least two disjoint  $u$ - $v$  paths in  $G$ . This implies the existence of a  $u$ - $v$  path in  $G$  of length at least 2. Among all  $u$ - $v$  paths in  $G$  choose one of maximum length, say

$$P: u = u_0, u_1, u_2, \dots, u_n = v \quad (n \geq 2).$$

Since  $G$  is connected there is a vertex  $x \in V(G) - V(P)$  which is adjacent to some  $u_i$ ,  $0 \leq i \leq n$ . In fact, we can assume  $0 < i < n$ ; for otherwise, each  $y \in V(G) - V(P)$  is adjacent to none of the vertices  $u_j$ ,  $0 < j < n$ , implying that

$G-u-v$  is disconnected (contradicting the 3-connectedness of  $G$ ). As  $G$  is locally 3-connected the subgraph  $\langle N(u_i) \rangle$  has an  $x-u_{i+1}$  path  $Q$  which contains neither  $u$  nor  $v$ . In addition, the path  $Q$  either does not contain the vertex  $u_{i-1}$  or it has an  $x-u_{i-1}$  subpath not containing  $u_{i+1}$ . There will be no loss in generality in assuming  $Q$  does not contain  $u_{i-1}$ . If  $x$  is adjacent to either  $u_{i-1}$  or  $u_{i+1}$ , then a  $u-v$  path of length  $n+1$  is easily seen to exist in  $G$ . Hence we assume  $x$  is adjacent to neither  $u_{i-1}$  nor  $u_{i+1}$ . Since the subgraph  $\langle \{x, u_{i-1}, u_i, u_{i+1}\} \rangle$  cannot be isomorphic to  $K(1, 3)$ ,  $u_{i-1}$  must be adjacent to  $u_{i+1}$ . Further, if  $V(P) \cap V(Q) = \{u_{i+1}\}$ , then again a  $u-v$  path of length greater than  $n$  exists in  $G$ . Thus we assume that  $V(P) \cap V(Q) \neq \{u_{i+1}\}$ .

A vertex  $w \in V(P) \cap V(Q) - \{u_{i+1}\}$  will be called *singular* if no vertex neighboring  $w$  on  $P$  is adjacent to  $u_i$ . Note that if  $w$  is a singular vertex and  $w_1$  and  $w_2$  are the vertices neighboring  $w$  on  $P$ , then  $w_1$  and  $w_2$  cannot be vertices of  $Q$ . Also, since the subgraph  $\langle \{w, u_i, w_1, w_2\} \rangle$  cannot be isomorphic to  $K(1, 3)$ ,  $w_1$  and  $w_2$  must be adjacent in  $G$ . The remainder of the proof proceeds by cases.

*Case 1. Each vertex of  $V(P) \cap V(Q) - \{u_{i+1}\}$  is singular.* From the preceding discussion it follows that if  $w \in V(P) \cap V(Q) - \{u_{i+1}\}$  and  $w_1$  and  $w_2$  are the vertices neighboring  $w$  on  $P$ , then  $w_1$  and  $w_2$  are adjacent in  $G$ . We now show the existence of a  $u-v$  path in  $G$  of length greater than  $n$ . Beginning at  $u$ , traverse  $P$  and for each  $u_j \in V(P) \cap V(Q)$ ,  $j < i$ , bypass  $u_j$  via the edge  $u_{j-1}u_{j+1}$ . Continue this process until  $u_i$  is reached. From  $u_i$  proceed to  $x$  and then along  $Q$  to  $u_{i+1}$ . Finally, from  $u_{i+1}$ , proceed along  $P$  and for each  $u_l \in V(P) \cap V(Q)$ ,  $l > i+2$ , bypass  $u_l$  via the edge  $u_{l-1}u_{l+1}$  until  $v$  is reached. The resulting  $u-v$  path contains each  $u_k$ ,  $0 \leq k \leq n$ , and contains  $x$ . This is a contradiction.

*Case 2.  $V(P) \cap V(Q) - \{u_{i+1}\}$  contains nonsingular vertices.* Let  $u_l$  be the first nonsingular vertex encountered if  $Q$  is traversed in the direction from  $x$  to  $u_{i+1}$ . Then either  $u_{l-1}$  or  $u_{l+1}$  is adjacent to  $u_i$ ; we assume without loss of generality that  $u_{l-1}$  is adjacent to  $u_i$ . We now replace  $P$  by a new  $u-v$  path  $P'$  of length  $n$ . If  $l < i$ , then we consider

$$P': u_0, u_1, \dots, u_{l-1}, u_i, u_l, u_{l+1}, \dots, u_{i-1}, u_{i+1}, u_{i+2}, \dots, u_n,$$

while if  $i < l$ , then we consider

$$P': u_0, u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_{l-1}, u_i, u_l, u_{l+1}, \dots, u_n.$$

In either case consider the  $x-u_l$  subpath  $Q'$  of  $Q$ . We note that  $V(P') \cap V(Q') - \{u_l\}$  consists entirely of singular vertices. Hence, by the argument given in Case

1, one can obtain a  $u$ - $v$  path in  $G$  of length greater than  $n$ , producing a contradiction.

This completes the proof. ■

We remark at this point that there exist graphs which are connected, locally connected, having no induced subgraph isomorphic to  $K(1, 3)$  and which are not hamiltonian-connected. For example, for each integer  $n \geq 3$ , the graphs  $K_2 + (K_{n-2} \cup K_2)$  have these properties.

It should be mentioned that the result just presented may not be best possible; that is, it is unknown whether Theorem 1 still follows if the condition "locally 3-connected" in the hypothesis is replaced by "locally 2-connected". With this altered hypothesis and the aid of Theorem B, the following result on 1-hamiltonian graphs (a graph  $G$  is 1-hamiltonian if it is hamiltonian and  $G-u$  is hamiltonian for each  $u \in V(G)$ ) can be quickly established.

**PROPOSITION 1.** *If  $G$  is a connected, locally 2-connected graph containing no induced subgraph isomorphic to  $K(1, 3)$ , then  $G$  is 1-hamiltonian.*

**PROOF.** By Theorem B, the graph  $G$  is hamiltonian. Let  $u \in V(G)$  and consider  $G-u$ . By Lemma 1,  $G$  is 2-connected. Thus, since  $G$  is locally 2-connected,  $G-u$  is both connected and locally connected. Hence by Theorem B,  $G-u$  is hamiltonian and the result follows. ■

We point out that Proposition 1 can be generalized; in fact, one can readily show that if  $G$  is a connected, locally  $(n+1)$ -connected graph containing no induced subgraph isomorphic to  $K(1, 3)$ , then  $G$  is  $n$ -hamiltonian.

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